A LOWER BOUND OF THE NUMBER OF EDGES IN A GRAPH CONTAINING NO TWO CYCLES OF THE SAME LENGTH

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Abstract

In 1975, P. Erdös proposed the problem of determining the maximum number $f(n)$ of edges in a graph of n vertices in which any two cycles are of different lengths. In this paper, it is proved that

$$f(n) \geq n + 32t - 1$$

for $t = 27720r + 169 \ \ (r \geq 1)$ and $n \geq \frac{6911}{16}t^2 + \frac{514441}{8}t - \frac{3309665}{16}$. Consequently, $\liminf_{n\to\infty} \frac{f(n)-n}{\sqrt{n}} \geq \sqrt{2 + \frac{2562}{6911}}$.

1 Introduction

Let $f(n)$ be the maximum number of edges in a graph on n vertices in which no two cycles have the same length. In 1975, Erdös raised the problem of determining $f(n)$ (see [1], p.247, Problem 11). Shi[2] proved that

$$f(n) \geq n + [(\sqrt{8n-23} + 1)/2]$$

for $n \geq 3$. Lai[3,4,5,6] proved that for $n \geq (1381/9)t^2 + (26/45)t + 98/45, t = 360q + 7$,

$$f(n) \geq n + 19t - 1,$$

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and for $n \geq e^{2m}(2m+3)/4$,

$$f(n) < n - 2 + \sqrt{nln(4n/(2m+3)) + 2n} + log_2(n+6).$$

Boros, Caro, Füredi and Yuster[7] proved that

$$f(n) \leq n + 1.98\sqrt{n}(1 + o(1)).$$

Let $v(G)$ denote the number of vertices, and $\epsilon(G)$ denote the number of edges. In this paper, we construct a graph G having no two cycles with the same length which leads to the following result.

Theorem. Let $t = 27720r + 169 \ \ (r \geq 1)$, then

$$f(n) \geq n + 32t - 1$$

for $n \geq \frac{6911}{16}t^2 + \frac{514441}{8}t - \frac{3309665}{16}$.

2 Proof of Theorem

Proof. Let $t = 27720r + 169, r \geq 1$, $n_t = \frac{6911}{16}t^2 + \frac{514441}{8}t - \frac{3309665}{16}$, $n \geq n_t$. We shall show that there exists a graph G on n vertices with $n + 32t - 1$ edges such that all cycles in G have distinct lengths.

Now we construct the graph G which consists of a number of subgraphs: B_i, $(0 \leq i \leq 21t + \frac{7t+1}{8} - 58, 22t - 798 \leq i \leq 22t + 64, 23t - 734 \leq i \leq 23t + 267, 24t - 531 \leq i \leq 24t + 57, 25t - 741 \leq i \leq 25t + 58, 26t - 740 \leq i \leq 26t + 57, 27t - 741 \leq i \leq 27t + 57, 28t - 741 \leq i \leq 28t + 52, 29t - 746 \leq i \leq 29t + 60, 30t - 738 \leq i \leq 30t + 60$, and $31t - 738 \leq i \leq 31t + 799)$.

Now we define these B_i's. These subgraphs all have a common vertex x, otherwise their vertex sets are pairwise disjoint.

For $\frac{7t+1}{8} \leq i \leq t - 742$, let the subgraph $B_{19t+2i+1}$ consist of a cycle

$$C_{19t+2i+1} = xx_i^1 x_i^2 ... x_i^{144t+13i+1463} x$$

and eleven paths sharing a common vertex x, the other end vertices are on the cycle $C_{19t+2i+1}$:

$$xx_{i,1}^1 x_{i,1}^2 ... x_{i,1}^{(11t-1)/2} x_i^{(31t-115)/2+i}$$

$$xx_{i,2}^1 x_{i,2}^2 ... x_{i,2}^{(13t-1)/2} x_i^{(51t-103)/2+2i}$$

$$xx_{i,3}^1 x_{i,3}^2 ... x_{i,3}^{(13t-1)/2} x_i^{(71t+315)/2+3i}$$

$$xx_{i,4}^1 x_{i,4}^2 ... x_{i,4}^{(15t-1)/2} x_i^{(91t+313)/2+4i}$$

$$xx_{i,5}^1 x_{i,5}^2 ... x_{i,5}^{(15t-1)/2} x_i^{(111t+313)/2+5i}$$

$$xx_{i,6}^1x_{i,6}^2...x_{i,6}^{(17t-1)/2}x_i^{(131t+311)/2+6i}$$

$$xx_{i,7}^1x_{i,7}^2...x_{i,7}^{(17t-1)/2}x_i^{(151t+309)/2+7i}$$

$$xx_{i,8}^1x_{i,8}^2...x_{i,8}^{(19t-1)/2}x_i^{(171t+297)/2+8i}$$

$$xx_{i,9}^1x_{i,9}^2...x_{i,9}^{(19t-1)/2}x_i^{(191t+301)/2+9i}$$

$$xx_{i,10}^1x_{i,10}^2...x_{i,10}^{(21t-1)/2}x_i^{(211t+305)/2+10i}$$

$$xx_{i,11}^1x_{i,11}^2...x_{i,11}^{(t-571)/2}x_i^{(251t+2357)/2+11i}.$$

From the construction, we notice that $B_{19t+2i+1}$ contains exactly seventy-eight cycles of lengths:

$21t+i-57$,	$22t+i+7$,	$23t+i+210$,	$24t+i$,
$25t+i+1$,	$26t+i$,	$27t+i$,	$28t+i-5$,
$29t+i+3$,	$30t+i+3$,	$31t+i+742$,	$19t+2i+1$,
$32t+2i-51$,	$32t+2i+216$,	$34t+2i+209$,	$34t+2i$,
$36t+2i$,	$36t+2i-1$,	$38t+2i-6$,	$38t+2i-3$,
$40t+2i+5$,	$40t+2i+744$,	$49t+3i+1312$,	$42t+3i+158$,
$43t+3i+215$,	$44t+3i+209$,	$45t+3i-1$,	$46t+3i-1$,
$47t+3i-7$,	$48t+3i-4$,	$49t+3i-1$,	$50t+3i+746$,
$58t+4i+1314$,	$53t+4i+157$,	$53t+4i+215$,	$55t+4i+208$,
$55t+4i-2$,	$57t+4i-7$,	$57t+4i-5$,	$59t+4i-2$,
$59t+4i+740$,	$68t+5i+1316$,	$63t+5i+157$,	$64t+5i+214$,
$65t+5i+207$,	$66t+5i-8$,	$67t+5i-5$,	$68t+5i-3$,
$69t+5i+739$,	$77t+6i+1310$,	$74t+6i+156$,	$74t+6i+213$,
$76t+6i+201$,	$76t+6i-6$,	$78t+6i-3$,	$78t+6i+738$,
$87t+7i+1309$,	$84t+7i+155$,	$85t+7i+207$,	$86t+7i+203$,
$87t+7i-4$,	$88t+7i+738$,	$96t+8i+1308$,	$95t+8i+149$,
$95t+8i+209$,	$97t+8i+205$,	$97t+8i+737$,	$106t+9i+1308$,
$105t+9i+151$,	$106t+9i+211$,	$107t+9i+946$,	$115t+10i+1307$,
$116t+10i+153$,	$116t+10i+952$,	$125t+11i+1516$,	$126t+11i+894$,
$134t+12i+1522$,	$144t+13i+1464$.		

Similarly, for $58 \leq i \leq \frac{7t-7}{8}$, let the subgraph $B_{21t+i-57}$ consist of a cycle

$$xy_i^1y_i^2...y_i^{126t+11i+893}x$$

and ten paths

$$xy_{i,1}^1y_{i,1}^2...y_{i,1}^{(11t-1)/2}y_i^{(31t-115)/2+i}$$

$$xy_{i,2}^1y_{i,2}^2...y_{i,2}^{(13t-1)/2}y_i^{(51t-103)/2+2i}$$

$$xy_{i,3}^1y_{i,3}^2...y_{i,3}^{(13t-1)/2}y_i^{(71t+315)/2+3i}$$

$$xy_{i,4}^{1}y_{i,4}^{2}\cdots y_{i,4}^{(15t-1)/2}y_i^{(91t+313)/2+4i}$$

$$xy_{i,5}^{1}y_{i,5}^{2}\cdots y_{i,5}^{(15t-1)/2}y_i^{(111t+313)/2+5i}$$

$$xy_{i,6}^{1}y_{i,6}^{2}\cdots y_{i,6}^{(17t-1)/2}y_i^{(131t+311)/2+6i}$$

$$xy_{i,7}^{1}y_{i,7}^{2}\cdots y_{i,7}^{(17t-1)/2}y_i^{(151t+309)/2+7i}$$

$$xy_{i,8}^{1}y_{i,8}^{2}\cdots y_{i,8}^{(19t-1)/2}y_i^{(171t+297)/2+8i}$$

$$xy_{i,9}^{1}y_{i,9}^{2}\cdots y_{i,9}^{(19t-1)/2}y_i^{(191t+301)/2+9i}$$

$$xy_{i,10}^{1}y_{i,10}^{2}\cdots y_{i,10}^{(21t-1)/2}y_i^{(211t+305)/2+10i}.$$

Based on the construction, $B_{21t+i-57}$ contains exactly sixty-six cycles of lengths:

$$\begin{array}{llll}
21t+i-57, & 22t+i+7, & 23t+i+210, & 24t+i, \\
25t+i+1, & 26t+i, & 27t+i, & 28t+i-5, \\
29t+i+3, & 30t+i+3, & 31t+i+742, & 32t+2i-51, \\
32t+2i+216, & 34t+2i+209, & 34t+2i, & 36t+2i, \\
36t+2i-1, & 38t+2i-6, & 38t+2i-3, & 40t+2i+5, \\
40t+2i+744, & 42t+3i+158, & 43t+3i+215, & 44t+3i+209, \\
45t+3i-1, & 46t+3i-1, & 47t+3i-7, & 48t+3i-4, \\
49t+3i-1, & 50t+3i+746, & 53t+4i+157, & 53t+4i+215, \\
55t+4i+208, & 55t+4i-2, & 57t+4i-7, & 57t+4i-5, \\
59t+4i-2, & 59t+4i+740, & 63t+5i+157, & 64t+5i+214, \\
65t+5i+207, & 66t+5i-8, & 67t+5i-5, & 68t+5i-3, \\
69t+5i+739, & 74t+6i+156, & 74t+6i+213, & 76t+6i+201, \\
76t+6i-6, & 78t+6i-3, & 78t+6i+738, & 84t+7i+155, \\
85t+7i+207, & 86t+7i+203, & 87t+7i-4, & 88t+7i+738, \\
95t+8i+149, & 95t+8i+209, & 97t+8i+205, & 97t+8i+737, \\
105t+9i+151, & 106t+9i+211, & 107t+9i+946, & 116t+10i+153, \\
116t+10i+952, & 126t+11i+894. & &
\end{array}$$

B_0 is a path with an end vertex x and length $n-n_t$. Other B_i is simply a cycle of length i.

It is easy to see that

$$\begin{aligned}
v(G) \;=\; & v(B_0)+\textstyle\sum_{i=1}^{19t+\frac{7t+1}{4}}(v(B_i)-1)+\sum_{i=\frac{7t+1}{8}}^{t-742}(v(B_{19t+2i+1})-1) \\
& +\textstyle\sum_{i=\frac{7t+1}{8}}^{t-742}(v(B_{19t+2i+2})-1)+\sum_{i=21t-1481}^{21t}(v(B_i)-1) \\
& +\textstyle\sum_{i=58}^{\frac{7t-7}{8}}(v(B_{21t+i-57})-1)+\sum_{i=22t-798}^{22t+64}(v(B_i)-1)+\sum_{i=23t-734}^{23t+267}(v(B_i)-1) \\
& +\textstyle\sum_{i=24t-531}^{24t+57}(v(B_i)-1)+\sum_{i=25t-741}^{25t+58}(v(B_i)-1)+\sum_{i=26t-740}^{26t+57}(v(B_i)-1) \\
& +\textstyle\sum_{i=27t-741}^{27t+57}(v(B_i)-1)+\sum_{i=28t-741}^{28t+52}(v(B_i)-1)+\sum_{i=29t-746}^{29t+60}(v(B_i)-1) \\
& +\textstyle\sum_{i=30t-738}^{30t+60}(v(B_i)-1)+\sum_{i=31t-738}^{31t+799}(v(B_i)-1)
\end{aligned}$$

$$
\begin{aligned}
&= n-n_t+1+\textstyle\sum_{i=1}^{19t+\frac{7t+1}{4}}(i-1)+\sum_{i=\frac{7t+1}{8}}^{t-742}(144t+13i+1463\\
&\quad +\tfrac{11t-1}{2}+\tfrac{13t-1}{2}+\tfrac{13t-1}{2}+\tfrac{15t-1}{2}+\tfrac{15t-1}{2}+\tfrac{17t-1}{2}+\tfrac{17t-1}{2}\\
&\quad +\tfrac{19t-1}{2}+\tfrac{19t-1}{2}+\tfrac{21t-1}{2}+\tfrac{t-571}{2})+\textstyle\sum_{i=\frac{7t+1}{8}}^{t-742}(19t+2i+1)\\
&\quad +\textstyle\sum_{i=21t-1481}^{21t}(i-1)+\sum_{i=58}^{\frac{7t-7}{8}}(126t+11i+893\\
&\quad +\tfrac{11t-1}{2}+\tfrac{13t-1}{2}+\tfrac{13t-1}{2}+\tfrac{15t-1}{2}+\tfrac{15t-1}{2}+\tfrac{17t-1}{2}+\tfrac{17t-1}{2}\\
&\quad +\tfrac{19t-1}{2}+\tfrac{19t-1}{2}+\tfrac{21t-1}{2})+\textstyle\sum_{i=22t-798}^{22t+64}(i-1)\\
&\quad +\textstyle\sum_{i=23t-734}^{23t+267}(i-1)+\sum_{i=24t-531}^{24t+57}(i-1)+\sum_{i=25t-741}^{25t+58}(i-1)\\
&\quad +\textstyle\sum_{i=26t-740}^{26t+57}(i-1)+\sum_{i=27t-741}^{27t+57}(i-1)+\sum_{i=28t-741}^{28t+52}(i-1)\\
&\quad +\textstyle\sum_{i=29t-746}^{29t+60}(i-1)+\sum_{i=30t-738}^{30t+60}(i-1)+\sum_{i=31t-738}^{31t+799}(i-1)\\
&= n-n_t+\tfrac{1}{16}(-3309665+1028882t+6911t^2)\\
&= n.
\end{aligned}
$$

Now we compute the number of edges of G

$$
\begin{aligned}
\epsilon(G) &= \epsilon(B_0)+\textstyle\sum_{i=1}^{19t+\frac{7t+1}{4}}\epsilon(B_i)+\sum_{i=\frac{7t+1}{8}}^{t-742}\epsilon(B_{19t+2i+1})\\
&\quad +\textstyle\sum_{i=\frac{7t+1}{8}}^{t-742}\epsilon(B_{19t+2i+2})+\sum_{i=21t-1481}^{21t}\epsilon(B_i)\\
&\quad +\textstyle\sum_{i=58}^{\frac{7t-7}{8}}\epsilon(B_{21t+i-57})+\sum_{i=22t-798}^{22t+64}\epsilon(B_i)+\sum_{i=23t-734}^{23t+267}\epsilon(B_i)\\
&\quad +\textstyle\sum_{i=24t-531}^{24t+57}\epsilon(B_i)+\sum_{i=25t-741}^{25t+58}\epsilon(B_i)+\sum_{i=26t-740}^{26t+57}\epsilon(B_i)\\
&\quad +\textstyle\sum_{i=27t-741}^{27t+57}\epsilon(B_i)+\sum_{i=28t-741}^{28t+52}\epsilon(B_i)+\sum_{i=29t-746}^{29t+60}\epsilon(B_i)\\
&\quad +\textstyle\sum_{i=30t-738}^{30t+60}\epsilon(B_i)+\sum_{i=31t-738}^{31t+799}\epsilon(B_i)\\
&= n-n_t+\textstyle\sum_{i=1}^{19t+\frac{7t+1}{4}}i+\sum_{i=\frac{7t+1}{8}}^{t-742}(144t+13i+1464\\
&\quad +\tfrac{11t+1}{2}+\tfrac{13t+1}{2}+\tfrac{13t+1}{2}+\tfrac{15t+1}{2}+\tfrac{15t+1}{2}+\tfrac{17t+1}{2}+\tfrac{17t+1}{2}\\
&\quad +\tfrac{19t+1}{2}+\tfrac{19t+1}{2}+\tfrac{21t+1}{2}+\tfrac{t-571+2}{2})+\textstyle\sum_{i=\frac{7t+1}{8}}^{t-742}(19t+2i+2)\\
&\quad +\textstyle\sum_{i=21t-1481}^{21t}i+\sum_{i=58}^{\frac{7t-7}{8}}(126t+11i+894\\
&\quad +\tfrac{11t+1}{2}+\tfrac{13t+1}{2}+\tfrac{13t+1}{2}+\tfrac{15t+1}{2}+\tfrac{15t+1}{2}+\tfrac{17t+1}{2}+\tfrac{17t+1}{2}\\
&\quad +\tfrac{19t+1}{2}+\tfrac{19t+1}{2}+\tfrac{21t+1}{2})+\textstyle\sum_{i=22t-798}^{22t+64}i\\
&\quad +\textstyle\sum_{i=23t-734}^{23t+267}i+\sum_{i=24t-531}^{24t+57}i+\sum_{i=25t-741}^{25t+58}i\\
&\quad +\textstyle\sum_{i=26t-740}^{26t+57}i+\sum_{i=27t-741}^{27t+57}i+\sum_{i=28t-741}^{28t+52}i\\
&\quad +\textstyle\sum_{i=29t-746}^{29t+60}i+\sum_{i=30t-738}^{30t+60}i+\sum_{i=31t-738}^{31t+799}i\\
&= n-n_t+\tfrac{1}{16}(-3309681+1029394t+6911t^2)\\
&= n+32t-1.
\end{aligned}
$$

Then $f(n)\geq n+32t-1$, for $n\geq n_t$. This completes the proof of the theorem.

From the above theorem, we have

$$\liminf_{n\to\infty}\frac{f(n)-n}{\sqrt{n}}\geq\sqrt{2+\frac{2562}{6911}},$$

which is better than the previous bounds $\sqrt{2}$ (see [2]), $\sqrt{2+\frac{487}{1381}}$ (see [6]).

Combining this with Boros, Caro, Füredi and Yuster's upper bound, we have

$$1.98\geq\limsup_{n\to\infty}\frac{f(n)-n}{\sqrt{n}}\geq\liminf_{n\to\infty}\frac{f(n)-n}{\sqrt{n}}\geq 1.5397.$$

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